Fall 2024

MATH3060 Mathematical Analysis III

Selected Solution to Mid-Term Examination

Answer any five of the following questions.

1. (a) (10 marks) Find the Fourier series of the function $f(x) = |x|$ (extended as a 2π-periodic function from $[-\pi, \pi]$).

   (b) (10 marks) Is this series convergent uniformly to f? Justify your answer.

2. Let f be a continuous 2π-periodic function and $f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$.

   (a) (10 marks) Establish the formula
   $$2\pi a_n = \int_{-\pi}^{\pi} [f(x) - f(x + \pi/n)] \cos nx \, dx, \quad n \geq 1.$$

   (b) (10 marks) Show that the Fourier coefficients of f decay to 0 as $n \to \infty$ using (a).

3. Let $S_n f$ be the n-th partial sum of the Fourier series of f, a 2π-periodic function integrable on $[-\pi, \pi]$.

   (a) (10 marks) Deduce the formula
   $$S_n f(x) = \int_{-\pi}^{\pi} D_n(z) f(x + z) \, dz,$$
   where $D_n(x)$ is given by
   $$D_n(x) = \begin{cases} \dfrac{\sin\left(n + \frac{1}{2}\right)x}{2\pi \sin \frac{1}{2}x}, & x \neq 0 \\ \dfrac{2n+1}{2\pi}, & x = 0. \end{cases}$$

(b) (10 marks) Show that

$$\int_0^\delta |D_n(z)|\,dz \to \infty, \quad n \to \infty,$$

for every $\delta > 0$.

4. Let $p \in [1, \infty)$.

(a) (10 marks) Show that the p-norms on $\mathbb{R}^n$ given by

$$\|\mathbf{x}\|_p = (\sum_{k=1}^n |x_k|^p)^{1/p},$$

where $\mathbf{x} = (x_1, \cdots, x_n)$, are equivalent.

(b) (10 marks) Discuss the equivalence of the p-norms on $C[0,1]$ given by

$$\left(\int_0^1 |f(x)|^p\,dx \right)^{1/p}.$$

Solution (b) By Holder's inequality, for $p < q$,

$$\int_0^1 |f|^p\,dx \le \left(\int_0^1 1^{1-p/q}\,dx \right)^{(q-p)/q} \left(\int_0^1 |f|^{pq/p}\,dx \right)^{q/p},$$

which implies $\|f\|_p \le \|f\|_q$, that is, the p-norm is weaker than the q-norm. On the other hand, we claim it is in fact strict. For, let $f_n(x) = n^2(1/n - x)$ on $[0, 1/n]$ and vanishes on $[1/n, 1]$. Then $\|f_n\|_1 = 1/2$ and $\|f_n\|_r = n^{1-1/r}/(r+1)^{1/r}$ which tends to ∞ as $n \to \infty$. It shows the r-norm, $r > 1$, is strictly stronger than the 1-norm. Now, given $1 \le p < q$, we define g_n and r by the relation $f_n = g_n^p$ and $r = q/p$. Then $\|g_n\|_p$ is constant and $\|g_n\|_q \to \infty$ as $n \to \infty$. We conclude that the p-norm is strictly weaker than the q-norm.

5. Consider $C[-1, 1]$ under the metric induced by the supnorm.

(a) (10 marks) Show that the set $A = \{f \in C[-1,1] : f^2(x) > e^{f(x)},\ x \in [-1,1]\}$ is an open set.

(b) (5 marks) Is $C^1[-1,1]$, regarded as a subspace in $C[-1,1]$, a closed set? Give a proof if yes, and a counterexample if no.

(c) (5 marks) Find the closure and interior of the set $E = \{f \in C[-1,1] : -3 < f(x) < 16,\ f(1/2) \neq 0,\ f(0) = 0\}$.

Solution (a) Let $F : C[-1,1] \to C[-1,1]$ be given by $F(f)(x) = f^2(x) - e^{f(x)}$. It is clear that F is continuous from $C[-1,1]$ to itself under the sup-norm (no need to prove this fact). On the other hand, the set $P = \{g \in C[-1,1] : g(x) > 0, \forall x \in [-1,1]\}$ is open in $C[-1,1]$ (also no need to prove this). It follows that $A = F^{-1}(P)$ is open since the preimage of an open set under a continuous map is open.

(b) Not nec true. For example, the sequence of C^1-functions $\{(x^2 + 1/n)^{1/2}\}$ converges uniformly to $|x|$. However, $|x|$ is in $C[-1,1]$ but not in $C^1[-1,1]$ (it is not differentiable at the origin).

(c) The closure of E is $\{f \in C[-1,1] :\ -3 \leq f(x) \leq 16, \forall x \in [-1,1], f(0) = 0\}$. Its interior is the empty set.

6. (a) (10 marks) Show that the only open and closed sets in (a,b), $a < b$, are the empty set and (a,b) itself. Here (a,b) is endowed with the Euclidean metric.

(b) (10 marks) Show that the only open and closed sets in $\mathbb{R}^n, n \geq 2$, are the empty set and $\mathbb{R}^n$ itself.

Solution (a) Assume that this open and closed set A is nonempty. We claim that it must be $\mathbb{R}$. Suppose not. Fix $z \in A$ and consider the set $B = \{x \in A : x \geq z\}$. Set $c = \sup B \leq b$. If $c = b$, $[z,b)$ belongs to A, good. Let us assume $c < b$. Sine A is closed, c belongs to A. However, A is open means for some $\varepsilon > 0$, $c + \varepsilon$ also belongs to B, contradicting the def of c. Hence $B = [z, b) \subset A$. Similarly, we prove $(a, z] \subset A$.

(b) Let A be a nonempty an open and closed set in $\mathbb{R}^n$. Let L be a straight line in $\mathbb{R}^n$. One readily shows that $A \cap L$ is open and closed in L. By (a), $A \cap L = L$ and the desired conclusion follows.